Assignment 11

This homework is due Friday April 22.

There are total 39 points in this assignment. 35 points is considered 100%. If you go over 35 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should exhibit your work and contain full proofs. Bare answers will not earn you much.

This assignment covers Sections 6.6, 7.2 (mostly 7.2) of Textbook.

Recall that $C_\rho(z_0)$ denotes the circle of radius ρ centered at z_0. $D_\rho(z_0)$ denotes the open disk of radius ρ centered at z_0. $\overline{D}_\rho(z_0)$ denotes the closed disk of radius ρ centered at z_0.

(1) [5pt] Let f be analytic in the disk $D_5(0)$ and suppose that $|f(z)| \leq 10$ for $z \in C_3(1)$.
  (a) Find a bound for $|f^{(4)}(1)|$. (*Hint:* Use Cauchy Inequalities.)
  (b) Find a bound for $|f^{(4)}(0)|$. (*Hint:* Use Cauchy Inequalities for the circle $C_2(0)$. To get a bound on $|f(z)|$ on $C_2(0)$, remember that $C_2(0) \subseteq \overline{D}_3(1)$ and use Maximum Modulus Principle.)

(2) [10pt] By computing derivatives, find the following Taylor series for the functions below and state where it is valid.
  (a) $\cos z$ at $z = \frac{\pi}{2}$.
  (b) $\sinh z$ at $z = \frac{\pi}{2} i$.
  (c) $\mathrm{Log}(1 - z)$ at $z = 0$.
  (d) $\frac{1}{z}$ at $z = 1$.
  (e) e^z at $z = 1$.

(3) [12pt] Using methods other than computing derivatives, find the Maclaurin series (=Taylor series at 0) for the following functions.
  (a) $\cos^3 z$. (*Hint:* $4\cos^3 z = \cos 3z + 3\cos z$.)
  (b) $\mathrm{Arctan}\, z$. (*Hint:* Integrate the Maclaurin series for $\frac{1}{1+z^2}$.)
  (c) $(z^2 + 1)\sin z$. (*Hint:* $(z^2 + 1)\sin z = z^2 \cos z + \sin z$.)
  (d) $f(z) = e^z \cos z$. (*Hint:* Express cos through the exponential, expand the brackets.)
  (e) $\frac{1}{(1-z^2)^2}$. (*Hint:* Do $\frac{1}{1-z^2}$ first. Then note $\frac{1}{(1-z^2)^2} = \frac{1}{2z}\left(\frac{1}{1-z^2}\right)'$.)

(4) [3pt] Let $f(z) = \frac{e^z - 1}{z}$ and set $f(0) = 1$. Explain why f is analytic at $z = 0$ and find the Maclaurin series for $f(z)$.

— see next page —

(5) [4pt] Let $f(z) = (1+z)^\beta = \exp(\beta \operatorname{Log}(1+z))$ be the principal branch of $(1+z)^\beta$, where β is a fixed complex number. Establish the validity for $z \in D_1(0)$ of the binomial expression

$$\begin{aligned}(1+z)^\beta &= 1 + \beta z + \frac{\beta(\beta-1)}{2!}z^2 + \frac{\beta(\beta-1)(\beta-3)}{3!}z^3 + \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{\beta(\beta-1)\cdots(\beta-n+1)}{n!} z^n.\end{aligned}$$

(*Hint:* Express a_n through derivatives. Remember that $((1+z)^\alpha)' = \alpha(1+z)^{\alpha-1}$.)

Comment. In this sense, the usual binomial formula holds for arbitrary, not just positive integer, β:

$$(1+z)^\beta = \sum_{n=0}^{\infty} \binom{\beta}{n} z^n.$$

(6) [5pt]

(a) Find terms up to z^4 in the Maclaurin series for $\frac{1}{1+\sin z}$ by substituting the Maclaurin series for $\sin z$ into the Maclaurin series for $\frac{1}{1+w}$.

(b) Find terms up to z^4 of Maclaurin series for $\cos(e^z - 1)$ by a similar method.